ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2013, Article ID ama0110, 9 pages
ISSN 2307-7743
http://scienceasia.asia

MUTUA'S METHOD: PARTICULAR POLYNOMIAL AND COEFFICIENTS OF PARTICULAR INTEGRALS OF THE FORM $Ax^n e^{cx}$

SAMUEL FELIX MUTUA

Abstract. In many problems involving solutions to ordinary differential equations, students and researchers in the field of applied mathematics are faced with challenges of obtaining the particular integrals. In particular, when the defined problem is formulated with non-homogeneous Ordinary Differential Equations with constant coefficients, Methods of Undetermined coefficients prove to be lengthy. Solutions to this will be reached by use of a polynomial, which we shall call particular polynomial whose values at given instants will be used to determine the coefficients of these particular integrals. We will define the polynomial $p_q(t)$ of degree q for $q \leq n$, n been the order of the given ODE, by $p_q(t) = (t - m_1)(t - m_2)...(t - m_{q-1})(t - m_q)$. m_s being the root of the characteristic function all not equal to c (in the exponential). If k are the number of roots all equal to c , then the particular integral is,

$$y_p = \frac{1}{k!P_{n-k}(c)} x^k e^{cx} \quad k = 0,1,2,..., n$$.

1.0 INTRODUCTION

Previously the methods of undetermined coefficients and variation of parameters have been used as techniques for solving these particular integrals. The former involves differentiating the assumed particular integral, substituting to the given ODE and then equating the LHS and RHS coefficients to solve for values of the coefficients. In the latter method, the constants are replaced with parameters (functions) which are varied using the given ODE to solve them. Use of the particular polynomial will simplify this process of obtaining such particular integrals. This will prove helpful as it will give results without having to go through the entire complex algorithms encountered in the past.

2.0 MUTUA'S METHOD

2.1 PRELIMINARIES

Consider the second order ODE

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}$$

Its characteristic function is

2010 Mathematics Subject Classification: 34C07.
Key words and phrases: Mutua's method, particular polynomial.

$m^2 + 3m + 2 = 0$

Which has roots $m_1 = -1$ and $m_2 = -2$ and the complementary function is

$y_c = Ae^{-x} + Be^{-2x}$

To solve for the particular integral using method of undetermined coefficients, since 2 is not a root we let

$y_p = Ae^{2x}$, $y'_p = 2^1 Ae^{2x}$, $y''_p = 2^2 Ae^{2x}$

Substituting back to the given ODE yields

$(2^2)Ae^{2x} + 3(2^1)Ae^{2x} + 2Ae^{2x} = 12Ae^{2x} = e^{2x}$ (RHS)

$\Rightarrow \quad A = \frac{1}{12}$

$\therefore \quad y_p = \frac{1}{12}e^{2x}$

Now if we critically look at the coefficient $\frac{1}{12}$, its denominator has a very close correlation with the value of characteristic function $f(m) = m^2 + 3m + 2$ at $m = 2$

$f(2) = 2^2 + 3(2) + 2 = 12$

$\therefore \ A = \frac{1}{12} = \frac{1}{f(2)}$

2.2 THE PARTICULAR POLYNOMIAL

Now, we consider a general linear ODE of order n with constant coefficients,

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = e^{cx} \tag{1}$$

Where $a_n \neq 0$ and c is a constant,

We have the characteristic function

$$f(m) = a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 \tag{2}$$

We define a polynomial $p_q(t)$ of degree q for $q \leq n$ by

$$p_q(t) = (t - m_1)(t - m_2)\dots(t - m_{q-1})(t - m_q) \tag{3}$$

Where $m_1, m_2, ..., m_q$ is a subset of the roots of $f(m)$ all $\neq c$.

We shall call this particular polynomial as it will be used to determine the particular integral.

CASE1: If all the roots of the characteristic function are not equal to c, we find that $f(c) \neq 0$

And therefore we take our particular polynomial to be

$$p(t) = a_n m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 \tag{4}$$

Now, the coefficient A of the particular integral, $y_p = Ae^{cx}$ becomes $A = \frac{1}{p(c)}$ i.e. to say that Particular integral $y_p = \frac{1}{p(c)} e^{cx}$ (5)

CASE2:

If $m_1 = m_2 = ... = m_k = c$ and $m_{k+1}, ..., m_n \neq c$

Then we realise that;

$$f(m) = (m - m_1)(m - m_2)...(m - m_k)(m - m_{k+1})...(m - m_n) \tag{6}$$

Yields $f(c) = 0$,Since the first k factors become zero.

In this case the coefficient of particular integral cannot be $\frac{1}{f(c)}$. Clearly it can be seen that e^{cx} will be absorbed in the complementary function.

Also, $xe^{cx}, x^2 e^{cx}, ..., x^{k-1} e^{cx}$ are contained in the complementary and they cannot form the particular integral.

In this case we assume a particular integral of the form $y_p = Ax^k e^{cx}$ (7)

Remark

From the differentiation of $x^k e^{cx}$ (by product rule) where $k \in z^+$ we realise that as the order of differentiation increases, in some of the terms the value of finite k diminishes and ultimately tends to zero. The first term with k =0 is in the k^{th} ordered derivative and will be of the form Be^{cx} where

$B = k!$,

It must be noted that if the method of undetermined coefficient was to be used the resulting k^{th} ordered derivative needs to be differentiated $(n - k)$ more times so that

upon substitution to the given ODE and equating the LHS and RHS coefficients particular integral is obtained.

Now;

When $k = 1$, e^{cx} is absorbed in the complementary function.

We take $y_p = Axe^{cx}$ and the particular polynomial as

$$p(m) = (m - m_2)(m - m_3)...(m - m_n) \qquad (8)$$

Where $p(c) \neq 0$

Using the knowledge of integration by parts (its reverse), we arrive at the following.

The coefficient A as $\frac{1}{1.p(c)}$

When $k = 2$, e^{cx}, xe^{cx} are absorbed in the complementary function.

We take $y_p = Ax^2e^{cx}$

$$P(m) = (m - m_3)(m - m_4)...(m - m_n) \qquad (9)$$

Once again $p(c) \neq 0$

The coefficient A becomes $A = \frac{1}{2.1.p(c)} = \frac{1}{2!\,p(c)}$

For $k = 3,4,..., q \leq n$

The corresponding coefficients of A are; $\frac{1}{3!P_{n-3}(c)}, \frac{1}{4!P_{n-4}(c)},...., \frac{1}{q!P_{n-q}(c)}$

And the particular integrals are $Ax^3e^{cx}, Ax^4e^{cx},...., Ax^{q-1}e^{cx}$, with A_s as defined above. In general, the particular integral is given by;

$$y_p = \frac{1}{k!P_{n-k}(c)} x^k e^{cx} \quad k = 0,1,2,..., n \qquad (10)$$

Equation (10) illustrates ***MUTUA'S METHOD*** of finding the particular integral.

Note: If the RHS of equation (1) is multiplied by a constant, then equation (10) is also multiplied by the same constant when evaluating the particular integral.

3.0 APPLICATION

The above method can be used in modelling phenomenon in which the governing mathematical equations are linear non-homogeneous Ordinary Differential Equation with constant coefficients, the RHS being an exponential function.

3.1 NUMERICAL TEST CASES

EXAMPLE 1

Solve $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = e^{3x}$

Solution

The characteristic function is $2m^2 + 5m - 3 = 2(m+3)(m - 1/2) = 0$

$m_1 = -3, m_2 = 1/2$

$\therefore$ Complementary function $y_c = Ae^{-3x} + Be^{1/2x}$

Here all roots are not equal to 3

$\therefore$ Particular polynomial $P(m) = 2m^2 + 5m - 3$, $P(3) = 2(3)^2 5(3) - 3 = 30$

Thus $y_p = \frac{1}{30}e^{3x}$

And the complete general solution of the given ODE is

$$y = y_c + y_p = Ae^{-3x} + Be^{1/2x} + \frac{1}{30}e^{3x}$$

EXAMPLE 2

Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$

The characteristic function is $m^3 - m^2 - 4m + 4 = (m-1)(m-2)(m+2) = 0$

$m_1 = 1, m_2 = 2, m_3 = -2$

$\therefore$ Complementary function $y_c = Ae^{x} + Be^{2x} + Ce^{-2x}$

Here $m_2 = 2 = c$

$\therefore$ Particular polynomial $P(m) = (m-1)(m+2)$, $P(2) = (2-1)(2+2) = 4$

Thus $y_p = \frac{1}{1!.4} xe^{2x} = \frac{1}{4} xe^{2x}$

And the complete general solution of the given ODE is

$$y = y_c + y_p = Ae^{x} + Be^{2x} + Ce^{-2x} + \frac{1}{4} xe^{2x}$$

EXAMPLE 3

Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + -y = 4e^{x}$

The characteristic function is $m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$

$m_1 = 1, m_2 = 1, m_3 = 1$

$\therefore$ Complementary function $y_c = Ae^{x} + Bxe^{x} + Cx^2e^{x}$

Here $m_1 = m_2 = m_3 = 1 = c$

$\therefore$ Particular polynomial $P(m) = 1$, $P(1) = 1$

Thus $y_p = \frac{4}{3!.1} x^3e^{x} = \frac{2}{3} x^3e^{x}$

And the complete general solution of the given ODE is

$$y = y_c + y_p = Ae^{x} + Bxe^{x} + Cx^2e^{x} + \frac{2}{3} x^3e^{x}$$

EXAMPLE 4

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-7x}$

Solution

The characteristic function is $m^2 + 2m + 5 = (m+1+2i)(m+1-2i) = 0$

$m_1 = -1-2i, m_2 = -1+2i$

$\therefore$ Complementary function $y_c = A\cos x + B\sin x$

Here all roots are not equal to -7

$\therefore$ Particular polynomial $P(m) = m^2 + 2m + 5$, $P(-7) = 2(-7)^2 + 2(-7) + 5 = 40$

Thus $y_p = \frac{1}{40}e^{-7x}$

And the complete general solution of the given ODE is

$$y = y_c + y_p = A\cos x + B\sin x + \frac{1}{40}e^{-7x}$$

Note: The above procedure can also be used when $c \notin Z$

EXAMPLE 5

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{-1/3x}$

Solution

The characteristic function is $m^2 - 5m + 6 = (m-3)(m-2) = 0$

$m_1 = 2, m_2 = 3$

$\therefore$ Complementary function $y_c = Ae^{2x} + Be^{3x}$

Here all roots are not equal to -1/3

$\therefore$ Particular polynomial $P(m) = m^2 - 5m + 6$, $P(-1/3) = (-1/3)^2 - 5(-1/3) + 6 = 70/9$

Thus $y_p = 2(\frac{9}{70})e^{-1/3x} = \frac{9}{35}e^{-1/3x}$

And the complete general solution of the given ODE is

$$y = y_c + y_p = Ae^{2x} + Be^{3x} + \frac{9}{35}e^{-1/3x}$$

COMPARISON OF MUTUA'S METHOD OF PARTICULAR POLYNOMIAL WITH THE METHOD OF UNDETERMINED COEFFICIENTS

Suppose we redo example 3 with the METHOD OF UNDETERMINED COEFFICIENTS

Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + -y = 4e^x$

The characteristic function is $m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$

$m_1 = 1, m_2 = 1, m_3 = 1$

$\therefore$ Complementary function $y_c = Ae^x + Bxe^x + Cx^2e^x$

To solve for the particular integral, we assume a particular integral of the form.

$$y_p = Ax^3e^x$$

Where A is a constant to be determined. Differentiating the above equation three times we get,

$$y'_p = 3Ax^2e^x + Ax^3e^x$$

$$y''_p = 6Axe^x + 3Ax^2e^x + 3Ax^2e^x + Ax^3e^x = 6Axe^x + 6Ax^2e^x + Ax^3e^x$$

$$y'''_p = 6Ae^x + 6Axe^x + 12Axe^x + 6Ax^2e^x + 3Ax^2e^x + Ax^3e^x = 6Ae^x + 18Axe^x + 9Ax^2e^x + Ax^3e^x$$

Substituting into the given ODE with have

$$(6Ae^x + 18Axe^x + 9Ax^2e^x + Ax^2e^x) - 3(6Axe^x + 6Ax^2e^x + Ax^3e^x) + 3(3Ax^2e^x + Ax^3) - Ax^3e^x = 4e^x$$

Or

$$6Ae^x = 4e^x$$

$$\therefore A = \frac{2}{3}$$

$$\Rightarrow y_p = \frac{2}{3}x^3e^x$$

And the complete general solution of the given ODE is

$$y = y_c + y_p = Ae^x + Bxe^x + Cx^2e^x + \frac{2}{3}x^3e^x$$

3.2 CONCLUSION

Clearly from the above example we conclude that ***MUTUA's** method* of particular polynomial arrives to the required particular integral of the form Ax^ne^{cx} faster than the already existing alternatives. In particular when the ODE in question is of high order a lot of time is spent on differentiation. Thus ***MUTUA'S** method* is of great significance since it simplifies the computation process involved in obtaining particular integrals. This is an important aspect to any mathematical formula.

3.3 ACKNOWLEDGMENT

I would like to thank my academic friends and students for their immense support and appreciation of the work I have done. Particularly, I express my deep gratitude to Prof. Hamadi Iddi Boga (Principal Taita Taveta University College) for motivating me with

variable inputs, Mr. Raymond Musyoka Munyao (student) and Mr. Ibare John Kamau (fellow student) who helped in proof reading and editing of this work .

REFERENCES

[1] Glyn James (2007): Modern Engineering Mathematics, 4th edition, Pearson Prentice Hall.

[2] K.A Stroud and Dexter J.Booth (2003): Advanced Engineering Mathematics, 4th edition, Palgrave Macmillan, 175 Fifth Avenue, New York.

[3] K.A Stroud and Dexter J.Booth (2007): Engineering Mathematics, 6th edition, Palgrave Macmillan, 175 Fifth Avenue, New York.

[4] K.S.Bhamra (2010): Partial Differential Equation, PHI Learning Private Limited, New Delphi.

[5] Schaum's (2009): Ordinary Differential Equation, McGraw-Hill Companies, United States of America.

[6] Thomas George B (2008): Thomas Calculus, 11th edition, New Delhi: Person Education.

SAMUEL FELIX MUTUA

is a 23 years old, 4th year student and a young potential Mathematician pursuing BSC: Mathematics and Computer Science, from

TAITA TAVETA UNIVERSITY COLLEGE, MATHEMATICS AND INFORMATICS DEPARTMENT, P.O BOX 635- 80300, VOI-KENYA